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TECHNICAL NOTE

A note on conjugate forced convection boundary-layer flow past a flat plate

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1. INTRODUCTION

IN PRACTICE, conjugate heat transfer problems are very important because interfacial boundary conditions are unknown. These problems have been the subject of many investigations beginning with Perelman [1] in 1961, who considered the convective heat transfer in the boundary-layer flow over a flat plate of finite thickness with a two-dimensional thermal condition in the plate. A one-dimensional approximation of the conduction process in a flat plate has been introduced by Luikov [2] and several other boundary-layer conjugate flows have been described by, for example, Martynenko and Sokovishin [3], Pozzi and Lupo [4], and Yu *et al.* [5].

The purpose of the present note is to re-examine the problem of conjugate forced convection flow over a flat plate of finite thickness by using a quite different solution technique. This method is based on a recent paper by Merkin and Pop [6], who showed that the parabolic nature of the equations governing the boundary-layer flows along a vertical flat plate can be fully exploited. This enables a parameter, originally introduced by Pozzi and Lupo [4], to be removed from the system of equations and boundary conditions and thus the solution only depends on the Prandtl number. The transformed energy equation is solved numerically based on a finite-difference approximation. Pozzi and Lupo [4] in their study have solved the problem by the method of series expansions: one series is valid for small values of x (coordinate along the plate) and a second series which is valid for large values of x. Further, these authors have shown that if the series for small values of x is suitably transformed, using a Padé transformation, it can describe accurately the solution in the entire domain of the flow. This solution method, even though of considerable interest, still requires a considerable amount of numerical work to find the solution of a large number of interrelated ODEs. In fact, the amount of numerical work required may well exceed that needed for a direct numerical solution of the complete problem.

2. ANALYSIS

The problem analysed here is such that an incompressible flow of velocity U_∞, temperature T_∞ and thermal conductivity k_f, passes over a flat plate of length l, thickness b and thermal conductivity k_s, as shown in Fig. 1. The plate is insulated at the leading edge and the lower surface is maintained at the constant temperature T_1 $(>T_\infty)$. If the axial conduction along the wall is negligible when compared with the normal conduction across the wall, which is consistent with boundary-layer theory, then the temperature profile in the wall is linear in $\bar{y}$, the coordinate normal to the plane of the plate, and is given by

$$T_s = T_w(\bar{x}) + \frac{T_w(\bar{x}) - T_1}{b}\bar{y}. \tag{1}$$

Here, $T_w(\bar{x})$ is the temperature at the solid–fluid interface and is determined by the solution of the forced flow convection problem.

Assuming that dissipation is negligible, then the boundary-layer equations for steady laminar flow over a flat plate in

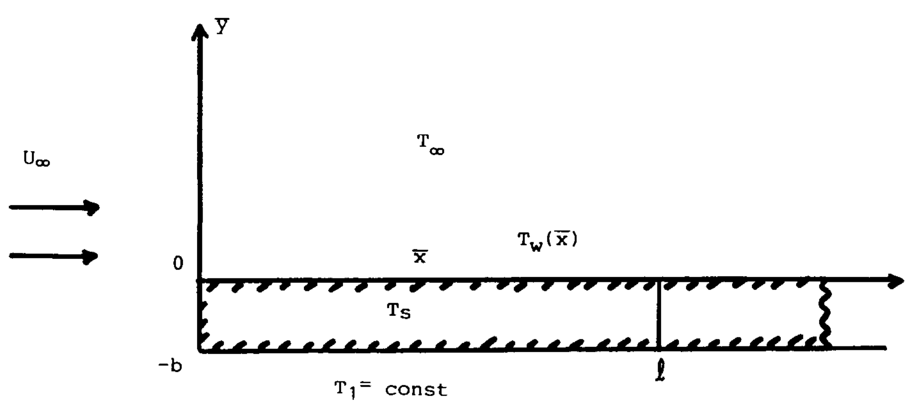

FIG. 1. Physical model and coordinate system.

NOMENCLATURE

b	thickness of the plate
f	scaled stream function, equation (6)
F	scaled temperature, equation (9a)
$g, \hat{g}$	scaled temperatures, equations (7a) and (8a)
k	thermal conductivity
l	length of the plate
L	length scale, equation (5a)
Pr	Prandtl number
Re	Reynolds number
T	temperature
T_1	temperature (constant) at the outer edge of the plate
U_∞	free stream velocity
x	coordinate along the plate
y	coordinate normal to the plate
Y	transformed coordinate, equation (9b).

Greek symbols

η	similarity variable
θ	non-dimensional temperature
ν	kinematic viscosity of the fluid
ξ	new variable, equation (9b)
ψ	stream function.

Superscripts

$'$	differentiation with respect to η
—	dimensional variables.

Subscripts

f	denotes fluid
s	denotes solid
w	condition at the solid–fluid interface
∞	condition in the free stream.

dimensionless form may be written as

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3} \tag{2}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{1}{Pr}\frac{\partial^2\theta}{\partial y^2} \tag{3}$$

which have to be solved subject to the boundary conditions

$$\psi=\frac{\partial\psi}{\partial y}=0,\quad \frac{\partial\theta}{\partial y}=\theta-1 \text{ on } y=0,\quad 0<x<\infty, \tag{4a}$$

$$\frac{\partial\psi}{\partial y}=1,\quad \theta=0 \text{ as } y\to\infty,\quad 0<x<\infty. \tag{4b}$$

In these equations the dimensionless variables are designed as

$$x=\bar{x}/L,\quad y=Re^{1/2}\bar{y}/L,\quad \psi=Re^{1/2}\bar{y}/(U_\infty L) \tag{5a}$$

$$\theta=(T-T_\infty)/(T_w-T_\infty),\quad L=\frac{U_\infty}{\nu}(bk_f/k_s)^2 \tag{5b}$$

where $Re=U_\infty L/\nu$ is the Reynolds number and L is the characteristic length of the plate, which is different to the choice made by Pozzi and Lupo [4] who used the plate thickness b as a reference length scale. The third boundary condition in (4a) has been obtained using equation (1) and the continuity of the heat flux condition at the interface, namely

$$k_s\frac{\partial T_s}{\partial\bar{y}}(\bar{x},0^-)=k_f\frac{\partial T}{\partial\bar{y}}(\bar{x},0^+).$$

It should be noted that the system of equations (2) and (3) involves only the single parameter Pr, the Prandtl number, as opposed to the non-dimensionalization used in ref. [4] which involves a further parameter which is related to the Brun number.

3. NUMERICAL SOLUTION

Equation (2) admits the well-known Blasius similarity solution

$$f(\eta)=\psi(x,y)/x^{1/2},\quad \eta=y/x^{1/2} \tag{6}$$

where the values of f and f' at each value of η may be found, for example, in ref. [7], and we now concentrate on the solution of equation (3). For small values of x, i.e. near the leading edge, this equation has a solution of the form

$$\theta=x^{1/2}g(x,\eta) \tag{7a}$$

and then boundary condition (4a) gives

$$g'=x^{1/2}g-1\quad \text{on } \eta=0. \tag{7b}$$

A solution can then be sought of the form

$$g(x,\eta)=\sum_{i=0}^{\infty}x^{i/2}g_i(\eta) \tag{8}$$

where the functions $g_i(\eta)$ must satisfy the differential equations and boundary conditions

$$\frac{1}{Pr}g_i''+\tfrac{1}{2}fg_i'-\tfrac{1}{2}(1+i)f'g_i=0,\quad i=0,1,2,\ldots \tag{9a}$$

$$\left.\begin{array}{ll} g_0'(0)=-1,\quad g_i'(0)=g_{i-1}(0) & i=1,2,3,\ldots \\ g_i(\infty)=0 & i=0,1,2,\ldots\end{array}\right\}. \tag{9b}$$

A solution which is valid for large values of x, i.e. far downstream, on the other hand, can be expressed in the form

$$\theta=\hat{g}(x,\eta) \tag{10a}$$

and now boundary condition (4a) becomes

$$x^{-1/2}\hat{g}'=\hat{g}-1\quad \text{on } \eta=0. \tag{10b}$$

The form of equation (10) suggests that the solution takes the form

$$\hat{g}(x,\eta)=\sum_{i=0}^{\infty}x^{-i/2}\hat{g}_i(\eta) \tag{11}$$

where the coefficient functions are given by the differential equations

$$\frac{1}{Pr}\hat{g}_i''+\tfrac{1}{2}f\hat{g}_i'+\tfrac{1}{2}f'\hat{g}_i=0,\quad i=0,1,2,\ldots \tag{12a}$$

and

$$\left.\begin{array}{ll} \hat{g}_0(0)=1,\quad \hat{g}_i(0)=\hat{g}_{i-1}'(0) & i=1,2,3,\ldots \\ \hat{g}_i(\infty)=0 & i=0,1,2,\ldots\end{array}\right\}. \tag{12b}$$

However, expansion (11) is not unique and eigensolutions of the form

$$\hat{g}(x,\eta)=\hat{g}_0(\eta)+x^{-\lambda}\hat{g}_\lambda(\eta) \tag{13}$$

exist where $\hat{g}_\lambda(\eta)$ satisfies

$$\frac{1}{Pr}\hat{g}_\lambda''+\tfrac{1}{2}f\hat{g}_\lambda'+\lambda f'\hat{g}_\lambda=0 \tag{14a}$$

$$\hat{g}_\lambda(0)=0,\quad \hat{g}_\lambda(\infty)=0. \tag{14b}$$

Table 1. Values of θ_w for $Pr = 0.7$†

ξ	Solution of equation (16)	Solution of equation (9): 21 terms	16 terms	11 terms	1 term	Solution (19)
0.02	0.022137	0.022137	0.022137	0.022137	0.022562	
0.129	0.12913	0.12913	0.12913	0.12913	0.14524	−4.01748
0.228	0.21044	0.21044	0.21044	0.21044	0.25734	−1.83178
0.346	0.29122	0.29122	0.29122	0.29122	0.39060	−0.86570
0.504	0.37843	0.37843	0.37843	0.37842	0.38302	−0.28238
0.659	0.44718	0.44718	0.44178	0.44693	0.74312	0.19355
0.814	0.50330	0.50331	0.50335	0.50091	0.91780	0.20614
1.094	0.58101	0.58171	0.58591	0.52589	0.12338	0.40937
1.414	0.64832	0.61269	0.86845	−0.19928	1.59482	0.54306
1.734	0.69767	−1.60878	5.98207	−6.55493	1.95580	0.62739
2.054	0.73521					0.68545
2.533	0.77709					0.76019
3.014	0.80767					0.78565
4.134	0.85465					0.84373
6.054	0.89785					0.89329
8.134	0.92284					0.92058
10.054	0.93709					0.93575

† Obtained from the present full numerical solution of equation (19) compared with the values obtained by solving equations (16) for small values of x and equations (12) for large values of x.

Table 2. Values of θ_w for $Pr = 7.02$†

ξ	Solution of equation (16)	Solution of equation (9): 21 terms	16 terms	11 terms	1 term	Solution (19)
0.02	0.047650	0.047650	0.047650	0.047650	0.049659	
0.1	0.20361	0.20361	0.20361	0.20345	0.24695	
0.2	0.34447	0.34447	0.34447	0.34447	0.49331	−0.46329
0.308	0.45306	0.45306	0.45306	0.45277	0.75810	0.04788
0.416	0.53380	0.53386	0.53406	0.52661	1.02609	0.29649
0.545	0.59913	0.60408	0.61815	0.47794	1.34328	0.46262
0.654	0.64245	0.61566	0.87690	−0.21789	1.61121	0.55198
0.815	0.69505	−2.27897	7.18659	−7.71475	1.99924	0.63893
1.020	0.74444					0.71525
2.099	0.86430					0.86041
3.059	0.90574					0.90422
4.019	0.92779					0.92710
6.099	0.95209					0.95196
8.019	0.96347					0.96346
9.939	0.97046					0.97052

† Obtained from the present full numerical solution of equation (16) compared with the values obtained by solving equations (9) for small values of x and equations (12) for large values of x.

Solutions of the transformed equations (9), (12) and (13) have been obtained, but this method of solution is not pursued further here as it was the method proposed by Pozzi and Lupo [4]. Here we are mainly concerned with the direct numerical integration of equation (3) by means of a finite-difference scheme, and to do this we use the method of continuous transformation proposed by Hunt and Wilks [8]. Hence, we write

$$\theta = \xi(1+\xi^2)^{-1/2}F(\xi, Y) \tag{15a}$$

where

$$Y = \xi^{-1}y, \quad \xi = x^{1/2} \tag{15b}$$

and note that (15) reduces to (7) for small values of x, and to (10) for large values of x.

On using (6) and (15), equation (3) becomes

$$\frac{1}{Pr}\frac{\partial^2 F}{\partial Y^2} + \tfrac{1}{2}f\frac{\partial F}{\partial Y} - \tfrac{1}{2}\frac{1}{1+\xi^2}f'F = \tfrac{1}{2}\xi f'\frac{\partial F}{\partial \xi} \tag{16}$$

which has to be solved subject to the boundary conditions

$$\frac{\partial F}{\partial Y} = \xi F-(1+\xi^2)^{1/2} \quad \text{on } Y = 0, \quad 0 < \xi < \infty \tag{17a}$$

$$F = 0 \quad \text{as } Y \to \infty, \quad 0 < \xi < \infty. \tag{17b}$$

The method used for solving equation (16), along with boundary conditions (17), to obtain the heat transfer characteristics of the flow is based on a finite-difference scheme that parallels that used by Merkin [9], and is therefore not reported here.

4. RESULTS AND DISCUSSION

Equations (9) have been solved numerically for $Pr = 0.7$ and 7.02 using a Runge–Kutta Merson method and the values of $g_i(0)$ agree with those given by Pozzi and Lupo [4] to within less than 0.1% for $i = 0, 1, \ldots, 19$, and are therefore not presented. It is of interest to note that as i increases, then $g_i(0)$ alternates in sign and increases in magnitude for $Pr = 0.7$, but decreases for $Pr = 7.02$, and thus the radius of convergence of the series will be larger for $Pr = 7.02$.

Numerical integration of equation (14) shows that the first eigensolution has an eigenvalue of $\lambda_1 = 0.80328$ for $Pr = 0.7$ and $\lambda_1 = 0.75777$ for $Pr = 7.02$, and therefore only the solution of equation (12) with $i = 0$ and 1 is sought. It is found that $\hat{g}_0' = -0.292680$ for $Pr = 0.7$ and $\hat{g}_0'(0) = -0.646542$ for $Pr = 7.02$. We note from equation (12) that $\hat{g}_1(\eta) = \hat{g}_0'(\eta)$ and thus $\hat{g}_1(0) = -0.292680$ for $Pr = 0.7$ and $\hat{g}_1(0) = -0.646542$ for $Pr = 7.02$.

The temperature of the wall of the plate is given by

$$\theta_w = \xi(1+\xi^2)^{-1/2}F(\xi, 0) \tag{18}$$

and its numerical value as a function of ξ is given in Tables 1 and 2 for $Pr = 0.7$ and 7.02, respectively. Also shown in these tables are the 21, 16, 11 and 1 term small ξ solutions and the large ξ solution, namely,

$$\left.\begin{array}{ll} \theta_w = 1-0.292680\xi^{-1} & Pr = 0.7 \\ \theta_w = 1-0.646542\xi^{-1} & Pr = 7.02 \end{array}\right\}. \tag{19}$$

For $Pr = 0.7$, it is observed that the 21 term small ξ solution agrees with the numerical solution to within about 4% up to $\xi = 0.65$ for 21 terms, up to $\xi = 0.55$ for 16 terms, up to $\xi = 0.4$ for 11 terms, up to $\xi = 0.3$ for 6 terms and up to $\xi = 0.02$ for 1 term. For large values of ξ, the large ξ solution is correct to within about 4% for $\xi \gtrsim 1$. No further terms in the large ξ solution can be easily calculated, as the next term in the expansion in equation (11) involves the eigensolutions.

The case of $Pr = 7.02$ is very similar to that for $Pr = 0.7$, except that the small ξ solution is valid over a much larger range of values of ξ, e.g. the 21 term solution agrees with the numerical solution to within about 4% up to $\xi = 1.4$, which compares with a value of $\xi = 0.65$ for $Pr = 0.7$. However, the large ξ solution agrees with the numerical solution to within about 4% for $\xi \geqslant 2.5$ which compares with $\xi \geqslant 1$ for $Pr = 0.7$.

In conclusion, in this paper we have shown how a simple transformation can be employed which reduces the number of parameters which occur in the problem by one in comparison to the techniques employed by previous researchers. Also we have illustrated that the asymptotic solutions which have been obtained by solving the sets of differential equations (9), (12) and (14) are very useful if solutions are only required for very small or very large values of x. If accurate solutions are required over the entire range of values of x, then an excessively large number of ordinary differential equations in the series for small values of x have to be obtained. In such circumstances it is much more efficient to solve the full governing parabolic partial differential equation (3) provided that the method of continuous transformation, as illustrated in equation (15), is employed.

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